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Matching extension and minimum degree

N. Ananchuen, L. Caccetta*

School of Mathematics and Statistics, Curtin University of Technology, GPO Box U1987, Perth, WA 6001, Australia

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Abstract

Let G be a simple connected graph on $2n$ vertices with a perfect matching. For a given positive integer k, $1 \leqslant k \leqslant n-1$, G is *k-extendable* if for every matching M of size k in G, there exists a perfect matching in G containing all the edges of M. The problem that arises is that of characterizing k-extendable graphs. In this paper, we establish a necessary condition, in terms of minimum degree, for k-extendable graphs. Further, we determine the set of realizable values for minimum degree of k-extendable graphs. In addition, we establish some results on bipartite graphs including a sufficient condition for a bipartite graph to be k-extendable.

1. Introduction

All graphs considered in this paper are finite, connected, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [4]. Thus, G is a graph with vertex set $V(G)$, edge set $E(G)$, $v(G)$ vertices, $\varepsilon(G)$ egdes, and minimum degree $\delta(G)$. For $V' \subseteq V(G)$, $G[V']$ denotes the subgraph induced by V'. Similarly, $G[E']$ denotes the subgraph induced by the edge set E' of G. $N_G(u)$ denotes the neighbour set of u in G and $\bar{N}_G(u)$ the nonneighbours of u. Note that $\bar{N}_G(u) = V(G) - N_G(u) - u$. The join $G \vee H$ of disjoint graphs G and H is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H.

A *matching* M in G is a subset of $E(G)$ in which no two edges have a vertex in common. M is a *maximum matching* if $|M| \geqslant |M'|$ for any other matching M' of G. A vertex v is *saturated* by M if some edge of M is incident to v; otherwise, v is said to be *unsaturated*. A matching M is *perfect* if it saturates every vertex of the graph. For simplicity we let $V(M)$ denote the vertex set of the subgraph $G[M]$ induced by M.

Let G be a simple connected graph on $2n$ vertices with a perfect matching. For a given positive integer k, $1 \leqslant k \leqslant n-1$, G is *k-extendable* if for every matching M of

* Corresponding author.

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size k in G, there exists a perfect matching in G containing all the edges of M. For convenience, a graph with a perfect matching is said to be 0-extendable. Observe that $K_{n,n}$ and K_{2n} are k-extendable for all k, $1 \leqslant k \leqslant n-1$. The cycle C_{2n} of order $2n$ is 1-extendable but not 2-extendable. A k-extendable graph G *k-minimal* if $G-e$ is not k-extendable for every edge e of G. Observe that C_{2n} is 1-minimal.

The problem that arises is that of characterizing k-extendable graphs. This problem has been the focus of considerable attention with many authors contributing results. An excellent survey is the paper of Plummer [10]. k-minimal graphs have been studied recently by Ananchuen and Caccetta [2, 3]. Two important results on k-extendable graphs due to Plummer [8], which are relevant to our work are:

Theorem 1.1. *Let G be a k-extendable graph on $2n$ vertices. $1 \leqslant k \leqslant n-1$. Then*
(a) *G is $(k-1)$-extendable;*
(b) *G is $(k+1)$-connected.*

Theorem 1.2. *Let G be a graph on $2n$ vertices and $1 \leqslant k \leqslant n-1$. If $\delta(G) \geqslant n+k$, then G is k-extendable.*

A consequence of Theorem 1.1(b) is that a k-extendable graph has minimum degree at least $k+1$. In Section 2, we prove that a k-extendable graph on $2n$ vertices has minimum degree at most n or at least $2k+1$. Further, we establish the existence of a k-extendable graph G on $2n$ vertices with $\delta(G) = j$ for every integer $j \in [k+1, n] \cup [2k+1, 2n-1]$.

In Section 3, we prove that a sufficient condition for a bipartite graph G on $2n$ vertices to be k-extendable is that $\delta(G) \geqslant \frac{1}{2}(n+k)$. Moreover, we establish the existence of a k-extendable bipartite graph G on $2n$ vertices with $\delta(G) = j$ for every integer $j \in [\frac{1}{2}(n+k), n]$. For k-minimal bipartite graphs, we establish an upper bound on the minimum degree.

2. Minimum degree of k-extendable graphs

In this section we establish a necessary condition, in terms of the minimum degree, for k-extendability. We make use of the following result, due to Ananchuen and Caccetta [1].

Theorem 2.1. *Let G be a k-extendable graph on $2n$ vertices with $\delta(G) = k+t$, $1 \leqslant t \leqslant k \leqslant n-1$. If $d_G(u) = \delta(G)$, then the subgraph $G[N_G(u)]$ has at most $t-1$ independent edges.*

We now prove the main result in this section.

Theorem 2.2. *If G is a k-extendable graph on $2n$ vertices, $1 \leqslant k \leqslant n-1$, then $k+1 \leqslant \delta(G) \leqslant n$ or $\delta(G) \geqslant 2k+1$.*

Proof. Suppose the assertion is false. Then, by Theorem 1.1(b), there exists a k-extendable graph G, $1 \leqslant k \leqslant n-1$, with $n+1 \leqslant \delta(G) \leqslant 2k$. Let $d_G(u) = r = \delta(G)$. By Theorem 2.1, the subgraph $H = G[N_G(u)]$ contains a maximum matching M with

$$|M| \leqslant r - k - 1 \leqslant k - 1.$$

Further, $|M| \geqslant 1$ as otherwise vertices of H have degree at most $2n - r < r$ in G.

Let F be a perfect matching in G containing M. Consider $M' = \{vw \in F \mid v, w \in \bar{N}_G(u)\}$. Clearly,

$$\begin{aligned} |M'| &= \tfrac{1}{2}\{(2n-r-1) - (r - 2|M| - 1)\} \\ &= n - r + |M| \\ &\leqslant |M| - 1 \\ &\leqslant k - 2. \end{aligned}$$

If $G[\bar{N}_G(u) \backslash V(M')]$ contains an edge e, then $M' \cup \{e\}$ is a matching of size at most $k-1$ which does not extend to a perfect matching in G, since $H - V(M)$ is an independent set of order $r - 2|M|$ and

$$\begin{aligned} |\bar{N}_G(u) \backslash V(M' \cup \{e\})| &= 2n - r - 1 - 2|M'| - 2 \\ &= 2n - r - 2(n - r + |M|) - 3 \\ &= r - 2|M| - 3. \end{aligned}$$

Hence, $G[\bar{N}_G(u) \backslash V(M')]$ has no edges.

Now, since $|M| \leqslant r - k - 1$,

$$|\bar{N}_G(u) \backslash V(M')| = r - 2|M| - 1 \geqslant 2k - r + 1 \geqslant 1.$$

Thus, there exists a vertex $x \in \bar{N}_G(u) \backslash V(M')$. If $xy \notin E(G)$ for all $y \in V(M)$, then

$$\begin{aligned} d_G(x) &\leqslant 2|M'| + r - 2|M| \\ &\leqslant 2(n-r) + r \\ &= 2n - r \\ &< r, \end{aligned}$$

a contradiction. So $xy \in E(G)$ for some $y \in V(M)$. Let $yy' \in M$ and consider $M'' = M' \cup \{xy, y'u\}$. M'' is a matching of size at most k that does not extend to a perfect matching in G, since

$$\begin{aligned} &|N_G(u) \backslash \{y, y'\}| - |\bar{N}_G(u) \backslash (V(M') \cup \{x\})| \\ &= (r-2) - (r - 2|M| - 2) \\ &= 2|M| \end{aligned}$$

and $H - y - y'$ has at most $|M| - 1$ independent edges. This contradicts the extendability of G and completes the proof of our theorem. □

We now consider the realizability problem associated with the above result. Let $\mathscr{G}(2n, k, j)$ denote the class of k-extendable graphs on $2n$ vertices with minimum degree j. We establish that $\mathscr{G}(2n, k, j) \neq \phi$ for every integer $j \in [k+1, n] \cup [2k+1, 2n-1]$. We begin with the case $2k + 1 \leqslant j \leqslant 2n - 1$.

Lemma 2.3. *For every integer* j, $2k + 1 \leqslant j \leqslant 2n - 1$, $\mathscr{G}(2n, k, j) \neq \phi$.

Proof. We distinguish two cases according to the parity of j.

Case 1: j odd.

Let $j = 2t + 1$, $k \leqslant t \leqslant n - 1$, and consider the graph $H_j = (n - t)K_2 \vee K_{2t}$. Clearly, $v(H_j) = 2n$, $\delta(H_j) = j$ and H_j has a perfect matching. We now prove that H_j is k-extendable. Let M be a matching of size k in H_j. It is convenient to write $H'_j = (n - t)K_2$ and $H''_j = K_{2t}$ so that $H_j = H'_j \vee H''_j$. Further, let $M = X \cup Y \cup Z$ where

$$X = \{ab \in M \mid a, b \in V(H'_j)\},$$

$$Y = \{ab \in M \mid a \in V(H'_j) \text{ and } b \in V(H''_j)\}, \text{ and}$$

$$Z = \{ab \in M \mid a, b \in V(H''_j)\}.$$

We denote the sets of M-unsaturated vertices of H'_j and H''_j by A and B respectively. Let A_i, $i = 0, 1$, denote the set of vertices of A having degree i in $H'_j - V(M)$. Fig. 1 illustrates our notation.

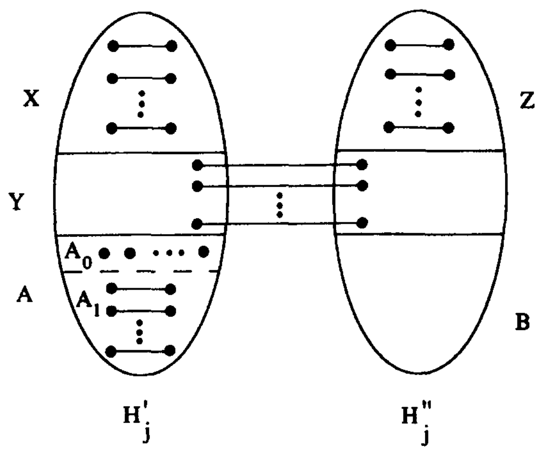

Fig. 1.

Clearly, $|A_0|$ and $|B|$ have the same parity. Further, $|A_0| \leqslant |Y|$ and

$$\begin{aligned}|B| &= 2t - 2|Z| - |Y| \\ &= 2t - 2(|Z| + |Y|) + |Y| \\ &= 2t - 2(k - |X|) + |Y| \quad (\text{since } k = |X| + |Y| + |Z|) \\ &= 2(t-k) + 2|X| + |Y| \\ &\geqslant 2|X| + |Y| \quad (\text{see } t \geqslant k) \\ &\geqslant |A_0|.\end{aligned}$$

The matching M can be extended to a perfect matching of G as follows. The edges joining vertices of A_1 form a matching M'. Let M'' be any set of $|A_0|$ independent edges joining vertices of A_0 and B. Note that, since $|A_0|$ and $|B|$ have the same parity, the graph $H_j - V(M \cup M' \cup M'')$ is a complete graph of even order and hence has a perfect matching M'''. Now $M \cup M' \cup M'' \cup M'''$ is the required perfect matching.

Case 2: j even.

Let $j = 2t$, $k+1 \leqslant t \leqslant n-1$. Consider the graph H_j obtained from the graph $(n-t)K_2 \vee K_{2t-1}$ by adding a vertex u and joining u to every vertex of K_{2t-1} and exactly one vertex of $(n-t)K_2$. Thus, $H_j = ((n-t-1)K_2 \cup P_3) \vee K_{2t-1}$ where P_3 is the path on 3 vertices. Clearly, $v(H_j) = 2n$, $\delta(H_j) = j$ and H_j has a perfect matching. For convenience we write $H'_j = (n-t-1)K_2 \cup P_3$ and $H''_j = K_{2t-1}$ so that $H_j = H'_j \vee H''_j$. We now prove that H_j is k-extendable.

Let M be a matching of size k in H_j. Define X, Y, Z, A and B as in Case 1. Further, let A_i, $i = 0, 1, 2$, denote the set of vertices of A having degree i in $H'_j - V(M)$. Fig. 2

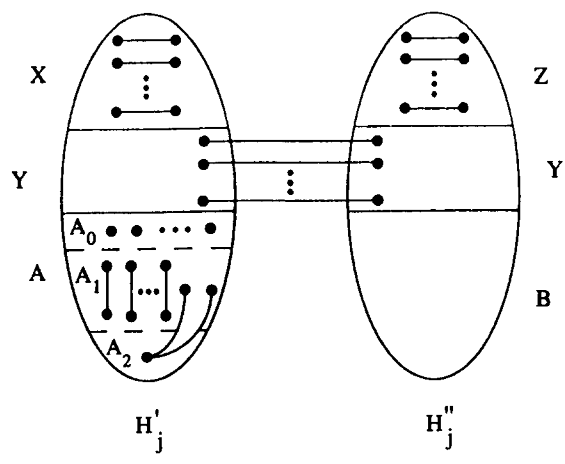

Fig. 2.

depicts the situation. Consider B. We have

$$\begin{aligned}|B| &= (2t-1) - 2|Z| - |Y| \\ &= (2t-1) - 2(|Z| + |Y|) + |Y| \\ &= (2t-1) - 2(k - |X|) + |Y| \quad (\text{since } k = |X| + |Y| + |Z|) \\ &= 2(t - k + |X|) + |Y| - 1.\end{aligned}$$

Thus, $|B|$ and $|Y|$ have different parity. Since $|A_1|$ is even and

$$|Y| = 2(n - t - |X|) - |A_1| - (|A_0| + |A_2|) + 1,$$

$|A_0| + |A_2|$ and $|Y|$ have different parity. Hence, $|B|$ and $|A_0| + |A_2|$ have the same parity. Now it is not too difficult to verify that $|A_0| \leqslant |Y| + 1$ and if $A_2 \neq \phi$, then $|A_0| \leqslant |Y|$. Since $t \geqslant k + 1$ and $|X| \geqslant 0$,

$$\begin{aligned}|B| &= 2(t - k + |X|) + |Y| - 1 \\ &\geqslant |Y| + 1.\end{aligned}$$

Now the matching M can be extended to a perfect matching of H_j as follows.

If $A_2 = \phi$, then we take M', M'' and M''' as defined in Case 1. So we need to consider only the case $A_2 \neq \phi$. Let $P_3 = u, v, w$. Clearly, $A_2 = \{v\}$ and $u, w \in A_1$. The edges joining vertice of $A_1 \setminus \{u, w\}$ together with the edge uv form a matching M' that saturates every vertex of $(A_1 \cup A_2) \setminus \{w\}$. Let M'' be any set of $|A_0| + 1$ independent edges joining vertices of $A_0 \cup \{w\}$ and B. The matching M'' is possible since $|A_0| \leqslant |Y|$ when $A_2 \neq \phi$ and $|B| \geqslant |Y| + 1 \geqslant |A_0| + 1$. Now the graph $H_j - V(M \cup M' \cup M'')$ is a complete graph of even order (as $|B|$ and $|A_0| + |A_2| = |A_0| + 1$ have the same parity) and hence has a perfect matching M'''. The matching $M \cup M' \cup M'' \cup M'''$ is the required matching. This completes the proof of our lemma. □

We now consider the range $k + 1 \leqslant j \leqslant n$.

Lemma 2.4. *For every integer j, $k + 1 \leqslant j \leqslant n$, $\mathscr{G}(2n, k, j) \neq \phi$.*

Proof. For $k + 1 \leqslant j \leqslant n$, define $G_0 = \bar{K}_{n-j} \vee \bar{K}_{j-1} \vee \bar{K}_j \vee \bar{K}_{n-j}$. Form the graph $\hat{G}$ from G_0 by adding a perfect matching between the vertices of the two $\bar{K}_{n-j}$'s. Observe that $\hat{G}$ has $2n - 1$ vertices and minimum degree j. We form the graph H_j from $\hat{G}$ by adding the vertex u and joining u to every vertex of $\bar{K}_j$. Fig. 3 illustrates the construction; for later reference we identify subgraphs G_1, G_2, G_3, and G_4 as indicated. Throughout the paper we adopt the convention that a double line in our diagram denotes the join between the corresponding graphs.

We will establish that $H_j \in \mathscr{G}(2n, k, j)$. As $\delta(H_j) = j$ we need only show that H_j is k-extendable. Observe that if $H_j - u - v$ is k-extendable for every $v \in V(G_3)$, then H_j is also k-extendable. Hence, it is sufficient to show that $H_j - u - v$ is k-extendable for every $v \in V(G_3)$.

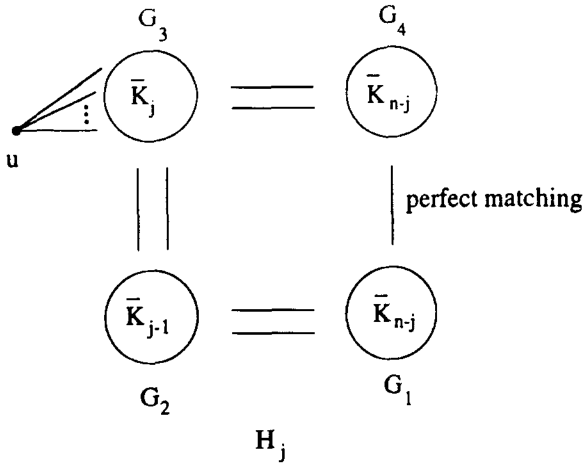

Fig. 3.

Let $v \in V(G_3)$ and consider the graph $H'_j = H_j - u - v$. Let M' be a matching in H'_j of size k. Further, let

$$M' = M_{12} \cup M_{23} \cup M_{34} \cup M_{41},$$

where M_{rs} denotes the edges of M' joining vertices of G_r to vertices of G_s, $1 \leqslant r \neq s \leqslant 4$.

Consider the subgraph $H'_j[(G_1 \cup G_4) - V(M')]$. Clearly, vertices of $G_1 \cup G_4$ have degree 0 or 1 in $H'_j[(G_1 \cup G_4) - V(M')]$. Let A_i and B_i, $i = 0, 1$, denote the vertices of $G_1 - V(M')$ and $G_4 - V(M')$, respectively having degree i in $H'_j[(G_1 \cup G_4) - V(M')]$. Then clearly $|A_1| = |B_1|$. Since, in H'_j, each vertex of A_0 (B_0) is joined to exactly one vertex of G_4 (G_1) incident to an edge of M_{34} (M_{12}), we have $|A_0| \leqslant |M_{34}|$ and $|B_0| \leqslant |M_{12}|$. Further,

$$|A_0| = (n - j) - |M_{12}| - |M_{41}| - |A_1|$$

and

$$|B_0| = (n - j) - |M_{34}| - |M_{41}| - |B_1|.$$

Thus,

$$|A_0| - |B_0| = |M_{34}| - |M_{12}| \tag{2.1}$$

Since $|M_{12}| + |M_{23}| + |M_{34}| + |M_{41}| = k$ and $j \geqslant k + 1$,

$$\begin{aligned} v(G_3 - (V(M') \cup \{v\})) &= (j - 1) - |M_{23}| - |M_{34}| \\ &= (j - 1) - k + |M_{12}| + |M_{41}| \\ &\geqslant |M_{12}| + |M_{41}| \\ &\geqslant |B_0| \quad (\text{since } |B_0| \leqslant |M_{12}|). \end{aligned}$$

Similarly,

$$\begin{aligned}
\nu(G_2 - V(M')) &= (j-1) - |M_{12}| - |M_{23}| \\
&= (j-1) - k + |M_{34}| + |M_{41}| \\
&\geqslant |M_{34}| + |M_{41}| \\
&\geqslant |A_0| \quad (\text{since } |A_0| \leqslant |M_{34}|).
\end{aligned}$$

Now the matching M' can be extended to a perfect matching of G as follows. Let M'_{41} be the set of edges joining vertices of A_1 and B_1. Further, let M'_{12} be the set of $|A_0|$ independent edges joining vertices of A_0 and $G_2 - V(M')$ and M'_{34} the set of $|B_0|$ independent edges joining vertices of B_0 and $G_3 - (V(M') \cup \{v\})$. M'_{12} and M'_{34} exist since $\nu(G_2 - V(M')) \geqslant |A_0|$ and $\nu(G_3 - (V(M') \cup \{v\})) \geqslant |B_0|$.

Now the graph $H'_j - V(M' \cup M'_{12} \cup M'_{34} \cup M'_{41})$ is a complete bipartite graph with bipartitioning sets $V(G'_2) = V(G_2) \backslash V(M' \cup M'_{12})$ and $V(G'_3) = V(G_3) \backslash (V(M' \cup M'_{34}) \cup \{v\})$. Consider $V(G'_2)$ and $V(G'_3)$. We have

$$\begin{aligned}
|V(G'_2)| &= (j-1) - |M_{12}| - |M_{23}| - |A_0| \\
&= (j-1) - |M_{12}| - |M_{23}| - (|M_{34}| - |M_{12}| + |B_0|), \quad \text{by (2.1)} \\
&= (j-1) - |M_{23}| - |M_{34}| - |B_0| \\
&= |V(G'_3)|.
\end{aligned}$$

Thus $H'_j - V(M' \cup M'_{12} \cup M'_{34} \cup M'_{41})$ has a perfect matching M'_{23}. Hence $M' \cup M'_{12} \cup M'_{23} \cup M'_{34} \cup M'_{41}$ is the required matching. This completes the proof of our lemma. □

Lemmas 2.3 and 2.4 yield the following theorem:

Theorem 2.5. *For every integer $j \in [k+1, n] \cup [2k+1, 2n-1]$, $\mathscr{G}(2n, k, j) \neq \phi$.*

Remark 2.1. For $n \geqslant 2k$, every integer in the interval $[k+1, 2n-1]$ is realizable as a minimum degree of a k-extendable graph on $2n$ vertices, $1 \leqslant k \leqslant n-1$. For $n \leqslant 2k-1$, this is not the case as no integer in the interval $[n+1, 2k]$ can be the minimum degree of a k-extendable graph on $2n$ vertices, $1 \leqslant k \leqslant n-1$.

3. Extendable bipartite graphs

The extendability properties of bipartite graphs have been studied by Little et al. [7], Plummer [9] and Brualdi and Csima [5]. These authors obtained some necessary and sufficient conditions for a bipartite graph to be k-extendable. In particular, Brualdi and Csima [5] proved:

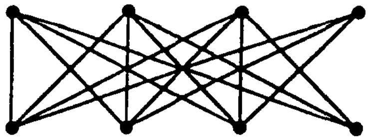

Fig. 4.

Theorem 3.1. *An r-regular bipartite graph of order 2n is k-extendable,* $2 \leqslant k \leqslant n-1$, *if and only if* $r = 1$ *or* $r \geqslant \frac{1}{2}(n+k)$.

In this section we establish that for sufficiency the 'regularity' condition in Theorem 3.1 can be replaced by 'minimum degree'. However, this is not the case for necessity as illustrated by the graph displayed in Fig. 4. Note that this nonregular graph is just the bipartite graph H_3 defined in the proof of Lemma 2.4 which is 2-extendable.

Before stating and proving our result we need to introduce some further terminology.

A *k-factor* of a graph G is a k-regular spanning subgraph of G. Katerinis [6] established the following sufficient condition for bipartite graphs to have a k-factor:

Theorem 3.2. *Let G be a bipartite graph with bipartitioning sets X and Y and k a positive integer. If*

(1) $|X| = |Y|$,

(2) $\delta(G) \geqslant \lceil |X|/2 \rceil \geqslant k$, *and*

(3) $|X| \geqslant 4k - 4\sqrt{k} + 1$ *when* $|X|$ *is odd and* $|X| \geqslant 4k - 2$ *when* $|X|$ *is even,*

then G has a k-factor.

We now establish our main result in this section.

Theorem 3.3. *If G is a bipartite graph with bipartitioning sets X and Y where* $|X| = |Y| = n$ *and* $\delta(G) \geqslant \frac{1}{2}(n+k)$, $1 \leqslant k \leqslant n-1$, *then G is k-extendable.*

Proof. By Theorem 3.2, G has a perfect matching. Let M be a matching of size k in G. Consider the graph $G' = G - V(M)$ with bipartitioning sets $X' = X \backslash V(M)$ and $Y' = Y \backslash V(M)$. Clearly, $|X'| = |Y'| = n - k$. If $k = n-1$, then $\delta(G) = n$ and thus $G \cong K_{n,n}$ which is $(n-1)$-extendable as required. So suppose that $1 \leqslant k \leqslant n-2$. Since $\delta(G') \geqslant \frac{1}{2}(n+k) - k = \frac{1}{2}(n-k) \geqslant 1$, Theorem 3.2 implies that G' has a perfect matching, as required. This completes the proof. □

Remark 3.1. The following construction shows that the bound on the minimum degree given in Theorem 3.3 is best possible. Consider the graph G displayed in Fig. 5

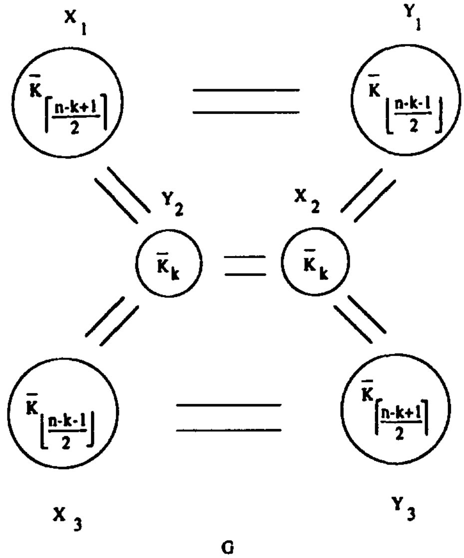

Fig. 5.

above. Clearly, G is a bipartite graph with bipartitioning sets $X = X_1 \cup X_2 \cup X_3$ and $Y = Y_1 \cup Y_2 \cup Y_3$. Further, $|X| = |Y| = n$ and $\delta(G) = \lfloor (n + k - 1)/2 \rfloor$. Now since $G - X_2 - Y_2$ is a bipartite graph without a perfect matching and the subgraph $G[X_2 \cup Y_2]$ is a $K_{k,k}$ and thus contains a matching of size k, G is not k-extendable. As our definition of k-extendability is for graphs with a 'perfect matching' we need to show that G has such a matching. Note that

$$1 \leqslant \lceil (n - k + 1)/2 \rceil - \lfloor (n - k - 1)/2 \rfloor \leqslant 2$$

and thus G has a perfect matching for every $k \geqslant 2$. In fact, only for $k = 1$ and $n - k$ even does G not have a perfect matching. In this case G has a maximum matching of size $n - 1$ with 2 unsaturated vertices, say x and y with $x \in X_1$ and $y \in Y_3$. But then the graph $G' = G + xy$ has a perfect matching, is bipartite and has $\delta(G') = \lfloor (n + k - 1)/2 \rfloor$. Further, G' is not k-extendable.

Remark 3.2. The graph H_j constructed in the proof of Lemma 2.4 is bipartite, k-extendable and $\delta(G) = j$, $k + 1 \leqslant j \leqslant n$. Consequently, there exists a k-extendable bipartite graph G on $2n$ vertices with $\delta(G) = j$ for every $j \in [\frac{1}{2}(n + k), n]$.

We conclude this paper with a discussion on k-minimal bipartite graphs. We make use of the following lemma proved in Ananchuen and Caccetta [2] in our work.

Lemma 3.4. *Let G be a k-extendable graph on $2n$ vertices, $1 \leqslant k \leqslant n - 1$. Then G is minimal if and only if for every edge $e = uv$ of G there exists a matching M of size k in*

$G - e$ such that $V(M) \cap \{u, v\} = \phi$ and for every perfect matching F, in G, containing $M, e \in F$.

Theorem 3.5. *If $G \neq K_{n,n}$ is a k-minimal bipartite graph on $2n$ vertices, $1 \leqslant k \leqslant n - 3$, then $\delta(G) < \frac{1}{2}(n + k)$.*

Proof. Suppose to the contrary that G is a k-minimal bipartite graph on $2n$ vertices, $1 \leqslant k \leqslant n - 3$ with $\delta(G) \geqslant \frac{1}{2}(n + k)$. Let X and Y be bipartitioning sets of G and $e = uv \in E(G)$ where $u \in X$, $v \in Y$. Clearly, $\delta(G - e) \geqslant \delta(G) - 1 \geqslant \frac{1}{2}(n + k) - 1$ and $G - e$ has the same bipartitioning sets as G.

If $\delta(G - e) > \frac{1}{2}(n + k) - 1$, then by Theorem 3.3, $G - e$ is k-extendable, contradicting the minimality of G. Hence $\delta(G - e) = \frac{1}{2}(n + k) - 1$. This implies that $n + k$ is even.

Since G is k-minimal, by Lemma 3.4, there exists a matching M of size k in $G - e$ such that $V(M) \cap \{u, v\} = \phi$ and e belongs to every perfect matching in G containing M. Let F be a perfect matching in G containing M. Consider the graph

$$G' = G - (V(M) \cup \{u, v\}).$$

Clearly,

$$F' = F \backslash (V(M) \cup \{u, v\})$$

is a perfect matching in G' and $\delta(G') \geqslant \frac{1}{2}(n - k) - 1$. Let

$$N_{G'}(u) = \{u_1, u_2, \dots, u_r\},$$

$$N_{G'}(v) = \{v_1, v_2, \dots, v_s\}.$$

Clearly $r, s \geqslant \frac{1}{2}(n - k) - 1$. If $u_i v_j \in F'$ for some $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$, then

$$F'' = (F \backslash \{uv, u_i v_j\} \cup \{uu_i, vv_j\}$$

is a perfect matching in $G - e$ containing M, contradicting the minimality of G. Hence, $u_i v_j \notin F'$ for all $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$. Now let

$$A = \{u_i'\colon u_i u_i' \in F',\ 1 \leqslant i \leqslant r\},$$

$$B = \{v_j'\colon v_j v_j' \in F',\ 1 \leqslant j \leqslant s\},$$

$$C = X \backslash (V(M) \cup N_{G'}(v) \cup A \cup \{u\}),$$

$$D = Y \backslash (V(M) \cup N_{G'}(u) \cup B \cup \{v\}).$$

Note that $A \cap N_{G'}(v) = B \cap N_{G'}(u) = \phi$ and $|C| = |D|$. Fig. 6 illustrates our notation; note that the edges of $M \cup \{e\}$ are indicated in bold whilst those of F' are indicated in wavy.

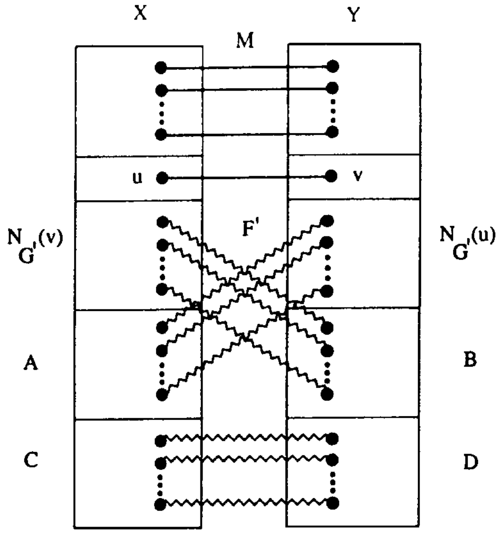

Fig. 6.

We claim that $u_i'w \notin E(G)$ for every $u_i' \in A$ and $w \in B \cup \{v\}$. Suppose this is not the case and $u_i'w \in E(G)$ for some i, $1 \leqslant i \leqslant r$, and $w \in B \cup \{v\}$. Then G contains the F-alternating path:

$$uu_iu_i'v, \quad \text{if } w = v$$

or

$$uu_iu_i'v_j'v_jv, \quad \text{if } w = v_j',$$

implying the existence of a perfect matching in $G - e$ containing M, a contradiction. Hence, $u_i'w \notin E(G)$ for every $u_i' \in A$ and $w \in B \cup \{v\}$. Similarly, $v_j'w \notin E(G)$ for every $v_j' \in B$ and $w \in A \cup \{u\}$. It follows that

$$\tfrac{1}{2}(n+k) \leqslant d_G(u_i') \leqslant n - s - 1, \quad \text{for } 1 \leqslant i \leqslant r,$$

and

$$\tfrac{1}{2}(n+k) \leqslant d_G(v_j') \leqslant n - r - 1, \quad \text{for } 1 \leqslant j \leqslant s.$$

Consequently, $r, s \leqslant \frac{1}{2}(n-k) - 1$. Hence since $r, s \geqslant \frac{1}{2}(n-k) - 1$ the only possibility is for $r = s = \frac{1}{2}(n-k) - 1 \geqslant 1$ (note that as $n+k$ is even, $n-k$ is even and is at least 4).

Consider the vertex $u_i' \in A$. We have $N_G(u_i') \subseteq Y \setminus (B \cup \{v\})$. Now, since

$$|Y| - |B \cup \{v\}| = n - 1 - \tfrac{1}{2}(n-k) + 1 = \tfrac{1}{2}(n+k),$$

we have $N_G(u'_i) = Y \backslash (B \cup \{v\})$ for every $u'_i \in A$. Similarly, $N_G(v'_j) = X \backslash (A \cup \{u\})$ for every $v'_j \in B$. Now

$$
\begin{aligned}
|C| = |D| &= n - k - 1 - |N_{G'}(u)| - |B| \\
&= n - k - 1 - 2s \\
&= 1.
\end{aligned}
$$

Let $C = \{c\}$ and $D = \{d\}$. Then G contains the edges cv'_j, $1 \leqslant j \leqslant s$ and du'_i, $1 \leqslant i \leqslant r$. But then

$$uu_1u'_1dcv'_1v_1v$$

is an F-alternating path in G and hence

$$F''' = (F \backslash \{uv, u_1u'_1, cd, v_1v'_1\}) \cup \{uu_1, u'_1d, cv'_1, v_1v\}$$

is a perfect matching in $G - e$ containing M, a contradiction. This completes the proof of the theorem. □

Remark 3.3. Theorem 3.5 is best possible in the sense that for $k = n - 2$, there exists an $(n-2)$-minimal bipartite graph with minimum degree $n - 1$; for example the $(n-1)$-regular bipartite graph.

Remark 3.4. In [2] we established that if $G \neq K_{2n}$ is a k-minimal graph on $2n$ vertices, then $\delta(G) \leqslant n + k - 1$.

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